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COMPLETE COHOMOLOGICAL FUNCTORS ON GROUPS*

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If Λ is a ring and A is a Λ-module, then a terminal completion of $\mathrm{Ext}^*_\Lambda(A,\ )$ is shown to exist if, and only if, $\mathrm{Ext}^j_\Lambda(A, P)=0$ for all projective Λ-modules P and all sufficiently large j. Such a terminal completion exists for every A if, and only if, the supremum of the injective lengths of all projective Λ-modules, silp Λ, is finite. Analogous results hold for $\mathrm{Ext}^*_\Lambda(\ , A)$ and involve spli Λ, the supremum of the projective lengths of the injective Λ-modules. When Λ is an integral group ring $\mathbb{Z}G$, spli $\mathbb{Z}G$ is finite implies silp $\mathbb{Z}G$ is finite. Also the finiteness of spli is preserved under group extensions. If G is a countable soluble group, the spli $\mathbb{Z}G$ is finite if, and only if, the Hirsch number of G is finite.

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The Tate cohomology of finite groups, and its generalization by Farrell to groups of finite virtual cohomological dimension, are both cohomological functors stretching from $-\infty$ to $+\infty$. Our first aim in this paper is to analyse what it means for a group to possess such complete cohomological functors.

We do this in Section 1 in a fairly abstract setting and find general criteria for the existence of cohomological completions. These lead naturally to two interesting types of homological finiteness conditions on a ring Λ: the supremum of the projective lengths (dimensions) of all injectives over Λ, which we call spli Λ, and the supremum of the injective lengths of all projectives, silp Λ.

Not much appears to be known about these homological dimensions. Results of Faith [5] and of Faith and Walker [6] show that silp $\Lambda=0$ if, and only if, spli $\Lambda=0$; and this occurs if, and only if, Λ is a quasi-Frobenius ring (meaning self-injective and Artinian). Jensen [9, 5.9] has shown that for a commutative noetherian ring Λ, silp Λ is finite if, and only if, spli Λ is finite. We prove here (Section 2) that for

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any Hopf algebra Λ over a suitable commutative ring K, if Λ is K-projective, then spli Λ finite implies silp Λ finite.

When Λ has finite global dimension m, then (almost obviously) silp Λ = spli $\Lambda = m$ (Corollary 1.7). This occurs in particular when Λ is the integral group ring of a group of finite cohomological dimension. It is an elementary consequence that if G has finite virtual cohomological dimension, then silp $\mathbb{Z}G$ = spli $\mathbb{Z}G$ is also finite (see 5.2). This is the connexion with Farrell cohomology.

Our main results on group rings are given in Section 5. They concern the behaviour of spli and silp under group extensions. We prove that the finiteness of spli is preserved under extensions (Theorem 5.5) and we generalise the group ring version of our Hopf algebra inequality to an extension theoretic result (Theorem 5.7). As an application of these theorems we show (Section 6) that if G is a soluble group with no non-trivial normal torsion subgroups, then silp $\mathbb{Z}G$ = spli $\mathbb{Z}G$ and this dimension is finite if, and only if, G has finite Hirsch number (torsion-free rank).

The heart of the paper is Sections 1 and 5; these sections may be read consecutively. Sections 2, 3, 4 should be viewed as appendices to Section 1 and Section 6 as an appendix to Section 5.

Some of our material overlaps with recent results of Bruce Ikenaga and we thank him for keeping us informed of his work. We have also benefited by advice from Maziar Shirvani and Bert Wehrfritz. We are especially grateful to Andreas Dress for very helpful critical comments on a first draft of this paper.

1. Complete cohomological functors

We begin by briefly recalling one version of the Tate–Farrell procedure (cf. [2] for details). Suppose H is a subgroup of finite index in the group G and A is a (right) $\mathbb{Z}G$-module such that $A\downarrow_H$, the restriction of A to H, has finite projective dimension k. Let P_* be a projective resolution of A over $\mathbb{Z}G$ and C be the image of P_k in P_{k-1}. Then $C\downarrow_H$ is projective and therefore the induced module $(C\downarrow_H)\otimes_{\mathbb{Z}H}\mathbb{Z}G$ is $\mathbb{Z}G$-projective. Let $G=\dot{\bigcup}_{t\in T} tH$ be a coset decomposition and define a map

$$\varphi:\ C\to(C\downarrow_H)\otimes_{\mathbb{Z}H}\mathbb{Z}G \quad \text{by } c\mapsto \sum_{t\in T} ct\otimes t^{-1}.$$

Then φ is a $\mathbb{Z}G$-homomorphism that splits over $\mathbb{Z}H$, whence the cokernel of φ is projective over $\mathbb{Z}H$. We may therefore repeat the construction with this cokernel replacing C and in this way produce a complex Q_*:

$$\cdots\to P_{k+1}\to P_k\to Q_{k-1}\to\cdots,$$

where

$$Q_{k-1}=(C\downarrow_H)\underset{\mathbb{Z}H}{\otimes}\mathbb{Z}G,\qquad Q_{k-2}=(\operatorname{coker}\varphi\downarrow_H)\underset{\mathbb{Z}H}{\otimes}\mathbb{Z}G,\ldots$$

(and $Q_i=P_i$ for $i\geq k$).

Now Q_* is a complete projective resolution of A in the sense that Q_* is a $\mathbb{Z}G$-projective acyclic complex coinciding with an ordinary projective resolution of A

above some dimension (here, above dimension k). The homology of $\mathrm{Hom}_G(Q_*, \ )$ provides a connected sequence of covariant functors stretching from $-\infty$ to $+\infty$ and coinciding with $\mathrm{Ext}^n_G(A, \ )$ for all $n > k$. When $A = \mathbb{Z}$, the sequence is independent of Q_* and is the Farrell cohomology of G.

We now turn to discuss cohomological functors abstractly. Let Λ be a given ring and $-\infty \leq r \leq s \leq +\infty$, where r and s are integers if they are finite. Suppose $T = (T^r, \ldots, T^s)$ is a sequence of covariant functors from Mod_Λ, the category of right Λ-modules, to abelian groups. (This notation needs obvious modification if one or both of r, s fail to be integers. For example, if $r = -\infty$ and s is finite, then T is understood to be a family of functors indexed by all $n \leq s$ and we may write $T = (\ldots, T^s)$.) Now T is a connected sequence of functors if there exist connecting homomorphisms δ so that an exact sequence in Mod_Λ,

$$0 \to A' \to A \to A'' \to 0,$$

gives rise to a complex of abelian groups,

$$T^r(A') \to T^r(A) \to T^r(A'') \to T^{r+1}(A') \to \cdots \to T^s(A''), \tag{1}$$

and this assignment is natural (in the sense that the assignment is a functor from the category of short exact sequences of Λ-modules to the category of complexes of abelian groups). (Cf. [4, III, § 4].) If (1) is exact, we say T is an *(r, s)-cohomological functor*; this is what Grothendieck [7, p. 140] calls an exact ∂-functor. Note that an (r, r)-cohomological functor is simply a half exact functor. We also remark that the generalities which follow remain true if Mod_Λ is replaced by any abelian category with enough projectives.

If $r \leq p \leq q \leq s$, then ${}_pT_q$ will denote the truncation $(T^p, \ldots, T^q)$ of T and we say T, or any functor naturally equivalent to T, is an *extension* of ${}_pT_q$. Obviously, if T is cohomological, ${}_pT_q$ is a (p, q)-cohomological functor.

A $(-\infty, +\infty)$-cohomological functor is called a *complete cohomological functor.* If U is a $(k, +\infty)$-cohomological functor (k an integer) and T is a complete cohomological functor which is an extension of ${}_nU_\infty$ for some $n \geq k$ (i.e., ${}_nU_\infty \cong {}_nT_\infty$ by a natural equivalence) then T is called a *completion* of U.

Example. In the Farrell situation discussed at the beginning of this section, $H^*(\mathrm{Hom}_G(Q, \ ))$ is a completion of $\mathrm{Ext}^*_G(A, \ )$.

A standard method of constructing cohomological functors is by satellites. Given a covariant functor F from Mod_Λ to abelian groups, the satellites S^nF, $n \in \mathbb{Z}$, form a $(-\infty, \infty)$-connected sequence of functors S^*F [4, III]. If F is half exact, S^*F is a complete cohomological functor. Moreover, $S^nF(P) = 0$ for all projective Λ-modules and all $n < 0$.

If U is a given (k, ∞)-cohomological functor and $m \geq k$, then because U^m is half exact, $(S^{j-m}U^m; j \leq m)$ gives $(-\infty, m)$-cohomological functor. If we hook this onto ${}_mU_\infty$, we obtain a completion of U.

Of course, there usually exist many other completions. For example, if U^k is left exact, we may extend U to all dimensions $<k$ with the zero functor. The distinguishing feature of the Farrell completion is that it is terminal. We explain this.

If R and T are completions of U, then a morphism $\varphi : R \to T$ is to be a natural transformation such that, for n large enough, ${}_n\varphi_\infty$ is the composite of the natural equivalences

$$ {}_nR_\infty \stackrel{\sim}{\to} {}_nU_\infty \stackrel{\sim}{\to} {}_nT_\infty . $$

The completion T is *terminal* if, for each completion R, there exists a unique morphism $R \to T$. A terminal completion is uniquely determined to within natural equivalence. This is proved in the same way as that a terminal object in a category is essentially unique. (In an appropriate set theoretic universe, the completions of U form a category and a terminal completion is a terminal object in this category.)

When can we expect a terminal completion of U to exist? The answer depends on the following results.

1.1 (Grothendieck [7, 2.2.1 and 2.2.2]). *Let T be an (r, s)-cohomological functor and let i be an integer satisfying $r \leqslant i \leqslant s$. The following assertions are equivalent*:

(i) $T^n(P) = 0$ *for all projective Λ-modules P and all $r \leqslant n < i$.*

(ii) *Given an (r, i)-cohomological functor R and a natural transformation $\varphi^i : R^i \to T^i$, then there exists one and only one extension of φ^i to $R \to {}_rT_i$.*

(iii) ${}_rT_i$ *is naturally equivalent to* $(S^{r-i}T^i, \ldots, S^{-1}T^i, T^i)$.

The implication (iii)$\Rightarrow$(i) comes from a property of satellites. The implication (i)$\Rightarrow$(ii) is essentially III, 5.2 in [4] and is well known. The implication (ii)$\Rightarrow$(iii) appears not widely known. For the convenience of the reader and because of its importance for what follows, we prove it here.

Assume (ii) and define R by $R^j = S^{j-i}T^i$, $r \leqslant j \leqslant i$. Then R is (r, i)-cohomological and so there exists a unique extension $\rho : R \to {}_rT_i$ of the identity on T^i. By (iii)$\Rightarrow$(ii), there exists a unique extension $\tau : {}_rT_i \to R$ of the identity on R^i. Then $\tau\rho = 1_T$ by hypothesis (ii) and $\rho\tau = 1_R$ by (iii)$\Rightarrow$(ii). Hence ${}_rT_i \cong R$, as required for (iii).

1.2. *The following statements about the $(-\infty, \infty)$-cohomological functor T are equivalent*:

(i) *T vanishes on projectives*;

(ii) *for each integer k, T is a terminal completion of ${}_kT_\infty$*;

(iii) *for each integer k, $T^j \cong S^{j-k}T^k$ for all $j \leqslant k$.*

Proof. (i)$\Rightarrow$(ii) comes from the corresponding implication in 1.1.

(ii)$\Rightarrow$(iii). Define $R^j = S^{j-k}T^k$ for all $j \leqslant k$ and hook ${}_{-\infty}R_k$ onto ${}_kT_\infty$ to produce a completion R of ${}_kT_\infty$. Since T is terminal, there exists a unique morphism $\rho : R \to T$ inducing a natural equivalence on ${}_lR_\infty$ for some $l \geqslant k$. By 1.1, (iii)$\Rightarrow$(ii), the identity on T^k extends uniquely to a natural transformation ${}_{-\infty}T_k \to {}_{-\infty}R_k$ and hence to a morphism of completions $\tau: T \to R$. Therefore $\tau\rho = 1_T$, because T is terminal. Since τ^l is the identity, ρ^l is also the identity and hence ${}_{-\infty}(\rho\tau)_l$ extends the identity. By 1.1, (iii)$\Rightarrow$(ii), ${}_{-\infty}(\rho\tau)_l$ is the identity, whence $\rho\tau = 1_R$. Consequently T is naturally equivalent to R, as required for (iii). □

We are now in a position to give a general criterion for the existence of a terminal completion.

1.3 Theorem. *Let U be a (k,∞)-cohomological functor. Then U has a terminal completion if, and only if, there exists $m \geq k$ such that $U^j(P) = 0$ for all projective Λ-modules P and all $j \geq m$.*

Proof. If a terminal completion exists, then by 1.2, U must vanish on projectives in all sufficiently high dimensions. Conversely, if U vanishes on projectives in all dimensions $\geq m$, then our satellite construction

$${}_mT_\infty = {}_mU_\infty, \qquad T^j = S^{j-m}U^m \quad \text{for } j \leq m,$$

produces a completion that vanishes on projectives. Hence T is terminal by 1.2. □

Our theorem has a curious consequence in case U is itself constructed by satellites, i.e., if $U^j = S^jU^k$ for $j \geq k$. If U has a terminal completion with natural equivalence from dimension m upwards, then by 1.2 (iii),

$$S^pS^qU^r \cong U^{p+q+r}$$

for all integers p, q (whether positive or negative) and all $r \geq k$ provided only that $q + r \geq m$ when $q < 0$ and $p + q + r \geq m$ when $p < 0$.

An example of such a functor U is $\mathrm{Ext}^*_\Lambda(A, \ )$. This is the functor to which we now exclusively turn our attention and for reference we restate Theorem 1.3.

1.4 Theorem. *Let A be a Λ-module. Then $\mathrm{Ext}^*_\Lambda(A, \ )$ has a terminal completion if, and only if, there exists $m(A) \geq 0$ so that $\mathrm{Ext}^j_\Lambda(A, P) = 0$ for all projective Λ-modules P and all $j \geq m(A)$.*

Example. In the Farrell situation, $\mathrm{Ext}^n_H(A, \ ) = 0$ for all $n > k$. *Since* $|G : H| < \infty$, the induction functor

$$M \to M_{\uparrow_H^G} = M \otimes_{\mathbb{Z}H} \mathbb{Z}G,$$

from $\mathrm{Mod}_{\mathbb{Z}H}$ to $\mathrm{Mod}_{\mathbb{Z}G}$, is naturally equivalent to the coinduction functor

$$M \to M^{\uparrow_H^G} = \mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, M).$$

Therefore, for any $\mathbb{Z}H$-projective module P and all $n > k$,

$$\mathrm{Ext}^n_G(A, P_{\uparrow_H^G}) = 0.$$

It follows from Theorem 1.4 that $\mathrm{Ext}^*_G(A, \ )$ has a terminal completion. In good situations this may be calculated from any complete projective resolution of A: cf. Section 4.

Suppose $\mathrm{Ext}^*_\Lambda(A,\ )$ has the terminal completion T. There is a useful description of T in terms of projective resolutions of the variable. If m is the integer $m(A)$ of Theorem 1.4, then we know

$$T^j = S^{j-m}\,\mathrm{Ext}^m_\Lambda(A,\ ) \quad \text{for all } j \in \mathbb{Z}.$$

Let B be a Λ-module, P_* a Λ-projective resolution of B, K_i the image of P_i in P_{i-1}, and $K_0 = B$. By the definition of satellites,

$$S^{-i}\,\mathrm{Ext}^m_\Lambda(A, B) \cong \mathrm{Ext}^m_\Lambda(A, K_i) \quad \text{for all } i \geqslant 0,$$

whence

$$T^j(B) \cong \mathrm{Ext}^m_\Lambda(A, K_{m-j}) \quad \text{for all } j \leqslant m. \tag{2}$$

The connecting homomorphism for T in dimensions $j \leqslant m$ can also be described in terms of resolutions. If

$$0 \to B' \to B \to B'' \to 0$$

is an exact sequence in Mod_Λ, choose projective resolutions P'_*, P''_* of B', B'', respectively and construct the associated resolution P_* of B [4, V, § 2]. Since

$$0 \to K'_i \to K_i \to K''_i \to 0$$

is exact, there is a connecting homomorphism

$$\delta\colon\ \mathrm{Ext}^m_\Lambda(A, K''_i) \to \mathrm{Ext}^{m+1}_\Lambda(A, K'_i);$$

and since $\mathrm{Ext}^m_\Lambda(A, P'_i) = 0$ for $i \geqslant m$, the connecting homomorphism

$$\delta'\colon\ \mathrm{Ext}^m_\Lambda(A, K'_i) \to \mathrm{Ext}^{m+1}_\Lambda(A, K'_{i+1})$$

is an isomorphism. Then

$$\delta(\delta')^{-1}\colon\ T^j(B'') \to T^{j+1}(B') \tag{3}$$

is the connecting homomorphism for T in dimensions $j \leqslant m$. The proof of this fact is straightforward (the naturality uses [4, XVII, 1.2]).

What does it mean if $\mathrm{Ext}^*_\Lambda(A,\ )$ possesses a terminal completion for every Λ-module A? By Theorem 1.4, each A determines a unique smallest integer $m(A)$. W claim that the set of these $m(A)$ is bounded above. Suppose not. Then for each $i > 0$ we can find a module A_i so that $m(A_i) > i$. Let $A = \bigoplus_{i \geqslant 1} A_i$ and $k = m(A)$. Now

$$\mathrm{Ext}^k_\Lambda(A,\ ) = \mathrm{Ext}^k_\Lambda(A_k,\ ) \oplus \mathrm{Ext}^k_\Lambda(A',\ ),$$

where $A_k \oplus A' = A$. Since $m(A_k) > k$, there exists a projective Λ-module P with $\mathrm{Ext}^k_\Lambda(A_k, P) \neq 0$, whence $\mathrm{Ext}^k_\Lambda(A, P) \neq 0$. This contradicts $m(A) = k$.

Let $d \geqslant m(A)$ for all modules A. Then $\operatorname{Ext}_\Lambda^n(\ , P)=0$ for all $n \geqslant d$ and every projective P. Thus every projective Λ-module has an injective resolution of length $<d$. We shall use the acronym silp Λ for the *supremum of the injective lengths of projective modules over* Λ; and may now state

1.5 Theorem. *The ring Λ has the property that* $\operatorname{Ext}_\Lambda^*(A, \ )$ *possesses a terminal completion for every Λ-module A if, and only if,* silp Λ *is finite.*

Bruce Ikenaga arrived from a somewhat different direction at a homological dimension very similar to silp [8]. In case $\Lambda = \mathbb{Z}G$, the difference between these two dimensions is at most 1.

There is a parallel theory to the above if one works with *contravariant* cohomological functors. For a fixed Λ-module B, $\operatorname{Ext}_\Lambda^*(\ , B)$ is such a functor in the interval $(0, \infty)$ and we have the following analogue of Theorem 1.4.

1.4′ Theorem. $\operatorname{Ext}_\Lambda^*(\ , B)$ *has a terminal completion if, and only if, there exists* $n(B) \geqslant 0$ *so that* $\operatorname{Ext}_\Lambda^j(I, B)=0$ *for all injective Λ-modules I and all* $j \geqslant n(B)$.

The analogue of Theorem 1.5 involves spli Λ, the *supremum of the projective lengths of the injective modules over* Λ.

1.5′ Theorem. *The ring Λ has the property that* $\operatorname{Ext}_\Lambda^*(\ , B)$ *possesses a terminal completion for every Λ-module B if, and only if,* spli Λ *is finite.*

The bifunctor $\operatorname{Ext}_\Lambda^*(\ , \ )$ is a bridge between these two theories. We make this point explicit in Section 3.

The relation between silp Λ and spli Λ remains unclear in general. Is it possible for the one dimension to be finite and the other infinite? Note however the almost trivial fact that if both dimensions are finite then they must be equal:

1.6. *If* silp Λ *and* spli Λ *are finite, then* silp Λ = spli Λ.

Proof. Suppose spli $\Lambda = n$ and let I be an injective Λ-module of projective dimension n. There exists B so that $\operatorname{Ext}_\Lambda^n(I, B) \neq 0$ and hence, if B is the image of the projective module P, $\operatorname{Ext}_\Lambda^n(I, P) \neq 0$. Thus silp $\Lambda \geqslant n$. A similar argument gives the opposite inequality.

1.7 Corollary. *If* gldim Λ, *the (right) global dimension of Λ, is finite, then*

$$\operatorname{gldim} \Lambda = \operatorname{silp} \Lambda = \operatorname{spli} \Lambda.$$

2. Hopf algebras

Our main result is Theorem 2.4. We begin with some preparatory lemmas.

Let K be a commutative ring and Λ a supplemented K-algebra. So we have ring homomorphisms $\varepsilon:\Lambda\to K$, $\eta:K\to\Lambda$ such that $\eta\varepsilon=1_K$. We shall assume Λ is K-projective.

2.1 Lemma. *If* spli Λ *is finite and* M *is a* Λ*-module of finite injective dimension over* K*, then* $\mathrm{Hom}_K(\Lambda, M)$ *has* Λ*-projective dimension* $\leqslant$ spli Λ.

Proof. Let I^* be a finite K-injective resolution of M as K-module. Then $J^k=\mathrm{Hom}_K(\Lambda, I^k)$ is Λ-injective and J^* is a Λ-resolution of $\mathrm{Hom}_K(\Lambda, M)$ (this uses the K-projectivity of Λ).

If spli $\Lambda\leqslant m$, $\mathrm{Ext}^r_\Lambda(J^k,\ )=0$ for all $r>m$, whence $\mathrm{Ext}^r_\Lambda(\mathrm{Hom}_K(\Lambda, M),\ )=0$ for all $r>m$, and consequently $\mathrm{Hom}_K(\Lambda, M)$ has projective dimension $\leqslant m$. □

2.2 Lemma. *Assume* K *has finite injective dimension* t *as* K*-module. If* A *is a* K*-projective* Λ*-module such that* $A\otimes_K D$ *has* Λ*-injective dimension* $\leqslant n$ *for every* K*-injective* D*, then* A *has* Λ*-injective dimension* $\leqslant n+t$.

Proof. Let I^* be a K-injective resolution of K of length t. Then $A\otimes_K I^*$ is an acyclic Λ-complex over A. Now

$$\mathrm{Ext}^r_\Lambda(\ , A\otimes I^k)=0 \quad \text{for all } r>n \text{ and all } k,$$

hence

$$\mathrm{Ext}^r_\Lambda(\ , A)=0 \quad \text{for all } r>n+t. \qquad \square$$

By a Hopf algebra Λ over K we mean a supplemented K-algebra satisfying the conditions (i)–(vi) of [4, XI, § 8]. (We remind the reader that we work in this paper with right modules; Cartan–Eilenberg set up their formulae for left modules.) So we have a K-algebra homomorphism $\Delta:\Lambda\to\Lambda\otimes\Lambda$ (we now abbreviate $\otimes_K$ as $\otimes$) and a K-algebra homomorphism $\omega:\Lambda\to\Lambda^\circ$, where Λ° is the opposite algebra to Λ. We shall denote by 1° the natural maps $\Lambda\to\Lambda^\circ$, $\Lambda^\circ\to\Lambda$ and write $\alpha^\circ=\alpha 1^\circ$ where α is in Λ or in Λ°. By a Λ-module we mean a module over Λ as K-algebra (*not* as Hopf algebra). If $\Lambda^e=\Lambda^\circ\otimes\Lambda$, the enveloping algebra of Λ, and M is a two-sided Λ-module, then M may be viewed as a right Λ^e-module by

$$m(\alpha^\circ\otimes\beta)=\alpha m\beta,$$

where $\alpha,\beta\in\Lambda$. Moreover, M becomes a right Λ-module by means of the algebra homomorphism $\Delta(\omega\otimes 1):\Lambda\to\Lambda^e$. Explicitly, if $\lambda\Delta=\Sigma\lambda'\otimes\lambda''$, then $m\lambda=\Sigma(\lambda'\omega)^\circ m\lambda''$.

2.3 Lemma. *Let* Λ *be a Hopf* K*-algebra. Suppose* A *is a* Λ*-module of* Λ*-projective dimension* $\leqslant m$ *and the* Λ*-module* B *is* K*-injective. Then* $\mathrm{Hom}_K(A, B)$ *is a* Λ*-module of* Λ*-injective dimension* $\leqslant m$.

Proof. Choose a Λ-projective resolution Q_* of A of length m. Since B is K-injective, $\mathrm{Hom}_K(Q_*, B)$ is a Λ-resolution of $\mathrm{Hom}_K(A, B)$ of dimension $\leqslant m$.

This complex is Λ-injective. To see this, in view of the Λ-projectivity of Q_*, it will suffice to verify that $\mathrm{Hom}_K(\Lambda, B)$ is Λ-injective. Here Λ acts diagonally: if $f \in \mathrm{Hom}_K(\Lambda, B)$ and $\lambda\Delta = \Sigma\lambda' \otimes \lambda''$, then

$$\alpha(f^\lambda) = \Sigma\alpha(\lambda'\omega)^\circ f\lambda''.$$

Let H denote the ordinary coinduced Λ-module $\mathrm{Hom}_K(\Lambda, B)$: if $f \in H$ and $\alpha, \lambda \in \Lambda$, then $\alpha(f^\lambda) = (\lambda\alpha)f$. The K-injectivity of B makes H Λ-injective. We shall be done if we can prove $\mathrm{Hom}_K(\Lambda, B)$ is Λ-isomorphic to H. To do this we define maps

$$\mathrm{Hom}_K(\Lambda, B) \rightleftarrows H$$

as follows: If f belongs to the left hand side, let $f^\rightarrow$ be the corresponding element in H, where

$$f^\rightarrow = \Delta(\omega \otimes 1)(1^\circ f \otimes 1)\pi$$

and $\pi : B \otimes \Lambda \to B$ is the map determined by the action of Λ on B, $b \otimes \lambda \mapsto b\lambda$. If f is an element of H, we let $f^\leftarrow$ be the element in the left hand side defined by exactly the same formula as was $f^\rightarrow$. It is easy to check that $f \mapsto f^\rightarrow$ is a Λ homomorphism but a little tricky to verify that $f^{\rightarrow\leftarrow} = f$, $f^{\leftarrow\rightarrow} = f$. Of course, only one of these need be checked. We sketch the argument for $f^{\rightarrow\leftarrow} = f$.

The conditions (i)-(vi) of [4, XI, § 8], will be referrred to as CE, (i)-(vi). Now

$$\begin{aligned} f^{\rightarrow\leftarrow} &= \Delta(\omega \otimes 1)(1^\circ f^\rightarrow \otimes 1)\pi \\ &= \Delta(\omega 1^\circ \Delta(\omega \otimes 1)(1^\circ f \otimes 1)\pi \otimes 1)\pi \end{aligned}$$

and

$$\omega 1^\circ \Delta = \Delta(\omega 1^\circ \otimes \omega 1^\circ) \quad (\text{CE}, (\text{v})),$$

$$\omega 1^\circ \omega = 1^\circ \quad (\text{CE}, (\text{iii})).$$

Hence

$$\begin{aligned} f^{\rightarrow\leftarrow} &= \Delta(\Delta \otimes 1)(1^\circ \otimes \omega 1^\circ \otimes 1)(1^\circ f \otimes 1 \otimes 1)(\pi \otimes 1)\pi \\ &= \Delta(1 \otimes \Delta)(f \otimes \omega 1^\circ \otimes 1)(\pi \otimes 1)\pi \quad (\text{CE}, (\text{iv})). \end{aligned}$$

If $\lambda\Delta = \Sigma\lambda' \otimes \lambda''$ and $\lambda'' = \Sigma\mu' \otimes \mu''$,

$$\begin{aligned} \lambda f^{\rightarrow\leftarrow} &= \Sigma(\lambda' f)(\mu'\omega 1^\circ)(\mu'') \\ &= \Sigma(\lambda' f)(\lambda''\varepsilon), \quad \text{by the commutative square on p. 194 of [4],} \\ &= (\Sigma\lambda'(\lambda''\varepsilon))f \\ &= \lambda f \ (\text{CE}, (\text{ii})), \quad \text{as required.} \end{aligned}$$

2.4 Theorem. *Let K be a commutative noetherian ring of finite K-injective dimension t. If Λ is a K-projective Hopf K-algebra, then*

$$\operatorname{silp} \Lambda \leqslant \operatorname{spli} \Lambda + t.$$

Proof. Without loss of generality we may assume spli Λ is finite, say m. Let P be a projective Λ-module. We must prove P has Λ-injective dimension $\leqslant m+t$. By Lemma 2.2 we need only show that for every injective K-module D, $P \otimes D$ has Λ-injective dimension $\leqslant m$. Write $R = P \otimes D$.

The augmentation ε of Λ yields the K-split injective Λ-homomorphism

$$\varepsilon' \colon\ K \to \operatorname{Hom}_K(\Lambda, K).$$

Writing $L = \operatorname{Hom}_K(\Lambda, K)$, ε' gives the K-split surjective Λ-homomorphism

$$\varepsilon'' \colon\ \operatorname{Hom}_K(L, R) \to R.$$

As R is relatively projective, ε'' is Λ-split. So it will suffice to prove $\operatorname{Hom}_K(L, R)$ has Λ-injective dimension $\leqslant m$.

This follows from Lemma 2.3. In the first place, L has Λ-projective dimension $\leqslant m$ by Lemma 2.1; secondly Λ is K projective and so $R = P \otimes D$, as K module, is a direct summand of a direct sum of copies of D, whence is K-injective since K is noetherian. Hence Lemma 2.3 applies and shows $\operatorname{Hom}_K(L, R)$ has Λ-injective dimension $\leqslant m$. $\square$

Note that the injective hypothesis on K in Theorem 2.4 is equivalent (since K is noetherian) to silp $K = t$ [9, 5.9]. A ring K as in Theorem 2.4 and of finite Krull dimension is called a Gorenstein ring.

3. Bifunctors

We now discuss briefly completions of Ext as a bifunctor. It is nearly obvious what one should mean by an (r, s)-Ext functor E: an exact, doubly connected sequence from r to s of functors in two variables (with variance as in Ext) and with anti-commuting connecting homomorphisms (cf. [4, V, § 4]). Explicitly, $E = (E^r, \ldots, E^s)$, where

(i) each E^i is a functor contravariant in a first variable and covariant in a second variable, from $\operatorname{Mod}_\Lambda \times \operatorname{Mod}_\Lambda$ to abelian groups;

(ii) the assignments $A \mapsto E(A,\ )$ and $A \mapsto E(\ , A)$ define functors from $\operatorname{Mod}_\Lambda$ to the category of (r, s)-cohomological functors on $\operatorname{Mod}_\Lambda$, with connecting homomorphisms δ_1, δ_2, respectively;

(iii) $\delta_1\delta_2 = -\delta_2\delta_1$.

Of course, $\mathrm{Ext}_\Lambda^*(\ ,\ )$ is a $(0,\infty)$-Ext functor in this sense. A $(-\infty,\infty)$-Ext functor is called complete and we define completions and terminal completions as expected.

Suppose silp Λ is finite, say silp $\Lambda = n-1$. Then for each Λ-module A we have the terminal completion T_A of $\mathrm{Ext}_\Lambda^*(A,\ )$ given by

$$_n(T_A)_\infty = {}_n\mathrm{Ext}_\Lambda^*(A,\ )_\infty,$$

$$T_A{}^j = S^{j-n}\,\mathrm{Ext}_\Lambda^n(A,\ ) \quad \text{for } j \leq n.$$

Clearly $T: A \to T_A$ is a contravariant functor from Mod_Λ to the category of complete cohomological functors on Mod_Λ. We claim that T can itself be made cohomological. So if

$$0 \to A' \to A \to A'' \to 0$$

is an exact sequence in Mod_Λ, we must define natural transformations

$$\delta:\ T_{A'}{}^j \to T_{A''}{}^{j+1}$$

to act as connecting homomorphisms. If $j \geq n$, δ is the connecting homomorphism in the first variable of Ext. Let $j < n$. The connecting homomorphism in the first variable yields

$$S^{j-n}\,\mathrm{Ext}_\Lambda^n(A'',\ ) \to S^{j-n}\,\mathrm{Ext}_\Lambda^{n+1}(A'',\ )$$

and 1.2(iii) shows that

$$S^{(j+1)-(n+1)}\,\mathrm{Ext}_\Lambda^{n+1}(A'',\ ) \rightrightarrows S^{j+1-n}\,\mathrm{Ext}_\Lambda^n(A'',\ ).$$

We take δ to be the composite of these natural transformations. The resulting long sequence of functors is exact and δ anti-commutes coherently with the connecting homomorphism δ_A on T_A (this depends on the anti-commutativity of the two differentials on Ext: [4, p. 85]).

Now T is a completion of $\mathrm{Ext}_\Lambda^*(\ ,\ )$. Moreover, T is terminal. This is an immediate consequence of the following version of 1.1, (i)$\Rightarrow$(ii).

3.1. *Let E be an (r,s)-*Ext *functor and i an integer such that $r \leq i < s$. If $E^n(A,P)=0$ for all A, all projective P, all $r \leq n \leq i$, and D is an $(r, i+1)$-*Ext *functor, then any natural transformation*

$$_iD_{i+1} \to {}_iE_{i+1}$$

extends in one and only one way to $D \to {}_rE_{i+1}$.

Note that the natural transformation is given at two consecutive dimensions. This is needed for the commutativity of

$$\begin{array}{ccc} E_{A'}{}^{j} & \longrightarrow & E_{A''}{}^{j+1} \\ \uparrow & & \uparrow \\ D_{A'}{}^{j} & \longrightarrow & D_{A''}{}^{j+1}. \end{array}$$

3.2. *If* silp Λ *is finite, then* $\mathrm{Ext}_\Lambda^*(\ ,\ )$ *has a terminal completion.*

Suppose next that spli Λ is finite. Then $\mathrm{Ext}_\Lambda^*(\ , B)$ has a terminal completion for every B and a parallel argument to the above yields

3.2′. *If* spli Λ *is finite, then* $\mathrm{Ext}_\Lambda^*(\ ,\ )$ *has a terminal completion.*

Of course, if silp Λ and spli Λ are both finite, the terminal completions provided by 3.2 and 3.2′ are naturally equivalent. If U is the terminal completion in this case, then $U(A, P)=0$ for all A and all projective P. Since each injective module has a finite projective resolution, we also have $U(A, I)=0$ for all A and all injective I. Similarly $U(Q, A)=0$ for all A and all projective or injective Q.

4. Complete resolutions

At the group-theoretic level, the theory of complete cohomological functors began with the idea of a complete projective resolution. We wish to make a few comments on the connexion of our work with such resolutions.

A complete projective resolution of a Λ-module A is an acyclic Λ-projective complex P_* which agrees with an ordinary projective resolution of A above some finite dimension. There is a similar definition for complete injective resolutions.

Suppose A has a complete projective resolution P_*. Then for all j, k,

$$H^j(\mathrm{Hom}_\Lambda(P_*, B)) \cong \mathrm{Ext}_\Lambda^k(K_{j-k}, B),$$

where K_r is the image of P_r in P_{r-1}. If silp Λ is finite, $\mathrm{Ext}_\Lambda^k(K_r, B)=0$ whenever B is projective and $k>\mathrm{silp}\,\Lambda$. It follows that $H^*(\mathrm{Hom}_\Lambda(P,\ ))$ is a terminal completion of $\mathrm{Ext}_\Lambda^*(A,\ \ )$ and thus is independent of the particular resolution P_*. This fact, but with a different proof, has also been noted by Ikenaga [8].

Whether the finiteness of silp necessarily implies that all or some modules possess complete projective resolutions we leave open. But we can state

4.1. *If* silp Λ *is finite, then every module has a complete injective resolution. If* spli Λ *is finite, then every module has a complete projective resolution.*

Proof. We only prove the first assertion; the second is handled similarly.

Suppose silp $\Lambda<n$. Let P_* be a projective resolution of A and construct an injective resolution $I^{*,*}$ of the complex P_* [4, XVII, § 1]. Thus $I^{*,k}$ is an injective

resolution of P_k. If $C^{i,k}$ is the kernel of $I^{i,k} \to I^{i+1,k}$, then $C^{n,k}$ is injective for all k, because silp $\Lambda < n$. Let J^r be the cokernel of $I^{r,1} \to I^{r,0}$. The diagram

$$\begin{tikzcd}
\cdots \arrow[r] & P_1 \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & I^{0,1} \arrow[r] \arrow[d] & I^{0,0} \arrow[r] \arrow[d] & J^0 \arrow[r] \arrow[d] & 0 \\
& \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
& I^{n-1,1} \arrow[r] \arrow[d] & I^{n-1,0} \arrow[r] \arrow[d] & J^{n-1} \arrow[r] \arrow[dr, tail] & 0 \\
& C^{n,1} \arrow[r] & C^{n,0} \arrow[rr, two heads] & & C \arrow[dl, tail] \\
& & & J^n &
\end{tikzcd}$$

enables us to read off the following complete injective resolution of A:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] & J^0 \arrow[r] & \cdots \arrow[r] & J^{n-1} \arrow[dr] & & & \\
& & \cdots \arrow[r] & C^{n,1} \arrow[r] & C^{n,0} \arrow[r] & J^n \arrow[r] & J^{n+1} \arrow[r] & \cdots .
\end{tikzcd}$$ □

Observe that, if both silp Λ and spli Λ are finite, then any complete projective or injective resolution can be used to calculate the terminal completion of $\mathrm{Ext}^*_\Lambda(\ ,\ )$ (cf. 3.2).

5. Group rings

We now return to group rings. Let K be a commutative ring, G a group and KG the group algebra of G over K. If H is a subgroup of G, we shall consider how terminal completions over KH and over $K(G/H)$ when H is normal, relate with those over KG. We begin with some straightforward facts.

Since the induction functor is left adjoint to the restriction functor, there is a natural isomorphism

$$\mathrm{Ext}^*_{KG}(A_{\uparrow_H^G}, B) \cong \mathrm{Ext}^*_{KH}(A, B\downarrow_H). \tag{4}$$

(Our notation for induction and coinduction is explained in the Example immediately following Theorem 1.4.) Coinduction is right adjoint to restriction and thus there is a natural isomorphism

$$\mathrm{Ext}^*_{KG}(B, A^{\uparrow_H^G}) \cong \mathrm{Ext}^*_{KH}(B\downarrow_H, A). \tag{5}$$

5.1. *Let A be a KH-module.*

(i) $\mathrm{Ext}^*_{KH}(A,\ )$ *has a terminal completion if, and only if,* $\mathrm{Ext}^*_{KG}(A_{\uparrow_H^G},\ )$ *has a terminal completion.*

(ii) $\mathrm{Ext}^*_{KH}(\ ,A)$ *has a terminal completion if, and only if,* $\mathrm{Ext}^*_{KG}(\ ,A^{\uparrow^G_H})$ *has a terminal completion.*

(iii) silp $KH \leqslant$ silp KG *and* spli $KH \leqslant$ spli KG.

Proof. (i) By Theorem 1.4, $\mathrm{Ext}^*_{KH}(A,\ )$ has a terminal completion if, and only if, there exists an integer m such that

$$\mathrm{Ext}^j_{KH}(A,P)=0 \quad \text{for all } j>m,$$

and all KH-projective modules P. Since P is a direct summand of $P_{\uparrow^G_H}\downarrow_H$, we may restrict attention to KH-projectives of the form $Q\downarrow_H$, where Q is KG-projective. Then relation (4) with $B=Q$ and Theorem 1.4 combine to complete the proof.

(ii) Theorem 1.4′ remains true if I is restricted to be cofree. But if C is KH-cofree, then $C^{\uparrow^G_H}$ is KG-cofree and C is a summand of $C^{\uparrow^G_H}\downarrow_H$. So we may complete the argument by relation (5) with $B=C$ and Theorem 1.4′.

(iii) The silp inequality is an immediate consequence of part (i) above and Theorem 1.5; the spli inequality comes from part (ii) and Theorem 1.5′. □

5.2. *Let $|G:H|$ be finite and M be a KG-module.*

(i) $\mathrm{Ext}^*_{KG}(M,\ )$ *has a terminal completion if, and only if,* $\mathrm{Ext}^*_{KH}(M\downarrow_H,\ )$ *has a terminal completion.*

(ii) $\mathrm{Ext}^*_{KG}(\ ,M)$ *has a terminal completion if, and only if,* $\mathrm{Ext}^*_{KH}(\ ,M\downarrow_H)$ *has a terminal completion.*

(iii) silp $KH =$ silp KG *and* spli $KH =$ spli KG.

Proof. Since induction is here naturally equivalent to coinduction, we have for any KH-module B,

$$\mathrm{Ext}^j_{KH}(M\downarrow_H, B)\cong \mathrm{Ext}^j_{KG}(M, B_{\uparrow^G_H})$$

and

$$\mathrm{Ext}^j_{KH}(B, M\downarrow_H)\cong \mathrm{Ext}^j_{KG}(B^{\uparrow^G_H}, M).$$

The rest of the argument is clear. □

For the next result recall that G is said to have virtual cohomological dimension n over K if there is a subgroup H of finite index in G so that K has projective dimension n as KH-module. This is independent of the choice of H. We write $\mathrm{vcd}_K G$ for the virtual cohomological dimension.

5.3 Corollary. *If* $\mathrm{vcd}_K G$ *and* gldim K *are finite, then*

$$\text{silp } KG = \text{spli } KG \leqslant \mathrm{vcd}_K G + \text{gldim } K.$$

Proof. Let H be a subgroup of finite index in G and of finite cohomological dimension over K. Then

$$\operatorname{gldim} KH \leqslant \operatorname{cd}_K H + \operatorname{gldim} K \quad [4, \text{XVI}, \S 4].$$

But

$$\begin{aligned}\operatorname{gldim} KH &= \operatorname{silp} KH = \operatorname{spli} KH \quad (\text{by Corollary 1.7})\\ &= \operatorname{silp} KG = \operatorname{spli} KG \quad (\text{by 5.2}). \qquad \square\end{aligned}$$

Suppose we are given a whole family (G_σ) of subgroups of G which are assumed to arise as the stabilizers of cells in an acyclic simplicial complex X on which G acts cellularly. We suppose (without any loss of generality) that X is an admissible G-complex (in the sense that the stabilizer G_σ of the cell σ fixes every point of σ). Given $\mathbb{Z}G$-modules A, B, the equivariant spectral sequence determined by G on X is

$$E_1^{p,q} = \prod_{\sigma \in \Sigma_p} \operatorname{Ext}^q_{G_\sigma}(A, B) \Rightarrow \operatorname{Ext}^{p+q}_G(A, B). \tag{6}$$

Here Σ_p is a transversal to the G-orbits on the p-cells of X [2, VII, § 5, § 7].

Suppose that $\operatorname{Ext}^*_{G_\sigma}(A, \ )$ has a terminal completion and define $m_\sigma = m(A\downarrow_{G_\sigma})$ (cf. Theorem 1.4). If X is finite dimensional and $\sup\{m_\sigma \mid \sigma \in \Sigma_p\} = m_p$ is finite for each p, then the spectral sequence shows that $\operatorname{Ext}^*_G(A, \ )$ has a terminal completion. Hence, if $\sup\{\operatorname{silp} \mathbb{Z}G_\sigma \mid \sigma \in \Sigma_p\}$ is finite for each p, so is $\operatorname{silp} \mathbb{Z}G$. Similar results hold for $\operatorname{Ext}^*_G(\ , B)$ and spli.

We apply these considerations to an interesting class of groups introduced by Ikenaga [8, p. 445]. Let $\mathfrak{C}_0$ be the class of all finite groups and inductively define $\mathfrak{C}_n$ from $\mathfrak{C}_{n-1}$ as follows: $G \in \mathfrak{C}_n$ if there exists an acyclic, finite dimensional admissible G-complex X for which each $G_\sigma \in \mathfrak{C}_{n-1}$ and $\sup\{\operatorname{silp} \mathbb{Z}G_\sigma \mid \sigma \in \Sigma\}$ is finite. (Here Σ is the union of all Σ_p, $p \geqslant 0$.) Then $\mathfrak{C}_{n-1} \subseteq \mathfrak{C}_n$ and we set $\mathfrak{C} = \bigcup \mathfrak{C}_n$.

By Serre's theorem [2, VIII, § 3], $\mathfrak{C}_1$ contains all groups of finite virtual cohomological dimension, but $\mathfrak{C}_1$ is a genuinely bigger class [8, p. 453, example 3].

5.4. *If* $G \in \mathfrak{C}$, *then* $\operatorname{silp} \mathbb{Z}G = \operatorname{spli} \mathbb{Z}G < \infty$.

The proof is an obvious induction on $\mathfrak{C}_n$, using the spectral sequence (6). The result is essentially in [8], as indeed are the appropriate parts of 5.1, 5.2 and Corollary 5.3. We remark that 5.4 and all considerations leading to it remain true when $\mathbb{Z}$ is replaced by any commutative ring of finite global dimension.

Ikenaga proves that for every group in $\mathfrak{C}$, there exists a complete projective resolution of $\mathbb{Z}$. He leaves open the question whether $\mathfrak{C}$ is extension closed. We shall show that this is indeed the case.

First however we study extensions with respect to the finiteness of silp and spli. For simplicity we restrict the coefficient ring to be $\mathbb{Z}$, though more general coefficient rings, such as Dedekind domains, could be used. We abbreviate $\operatorname{spli} \mathbb{Z}G$, $\operatorname{silp} \mathbb{Z}G$ as spli G, silp G, respectively, and write $\otimes$ for $\otimes_{\mathbb{Z}}$, $\operatorname{Hom}(\ ,\ )$ for $\operatorname{Hom}_{\mathbb{Z}}(\ ,\ )$.

5.5 Theorem. *If* $1 \to N \to E \to G \to 1$ *is an exact sequence of groups, then* spli $E \leqslant$ spli $N +$ spli G.

5.6 Lemma. *If A is a $\mathbb{Z}$-module and* spli $G = m$, *then the coinduced module $A^{\uparrow} = A^{\uparrow_1^G}$ has projective dimension $\leqslant m$.*

Proof. If A is embedded in an injective $\mathbb{Z}$-module D_0, then the cokernel D_1 is also injective. Apply the exact functor $(\ )^{\uparrow}$ and obtain

$$0 \to A^{\uparrow} \to D_0^{\uparrow} \to D_1^{\uparrow} \to 0,$$

a $\mathbb{Z}G$-injective resolution of $A^{\uparrow}$. Since $\operatorname{Ext}^j_G(D_i^{\uparrow},\ ) = 0$ for all $j > m$ and $i = 0, 1$, so $\operatorname{Ext}^j_G(A^{\uparrow},\ ) = 0$ for all $j > m$, as required.

Proof of Theorem 5.5. We may suppose spli $N = n$ and spli $G = m$ are finite. Choose an injective $\mathbb{Z}E$-module I. We must prove that I has projective dimension $\leqslant m + n$.

The augmentation sequence

$$0 \to \mathfrak{g} \to \mathbb{Z}G \to \mathbb{Z} \to 0$$

yields the $\mathbb{Z}$-split sequence of $\mathbb{Z}G$-modules

$$0 \to \mathbb{Z} \to J \to C \to 0, \tag{7}$$

where $J = \operatorname{Hom}(\mathbb{Z}G, \mathbb{Z})$ and $C = \operatorname{Hom}(\mathfrak{g}, \mathbb{Z})$. Hence I is a direct summand of $I \otimes J$ and so it will suffice to show that $I \otimes J$ has projective dimension $\leqslant m + n$.

By Lemma 5.6, since J is coinduced, J has $\mathbb{Z}G$-projective dimension $\leqslant m$. Let Q_* be a $\mathbb{Z}G$-projective resolution of length m.

Choose a $\mathbb{Z}E$-projective resolution of I and call P_* its truncation at dimension n:

$$0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to I \to 0.$$

Thus P_* is a $\mathbb{Z}E$-resolution of I, P_i is $\mathbb{Z}E$-projective for $i = 0, \ldots, n-1$ and P_n is $\mathbb{Z}N$-projective. It follows that (the total complex) $P_* \otimes Q_*$ is a $\mathbb{Z}E$-complex over $I \otimes J$ of dimension $\leqslant m + n$. Since J *is* $\mathbb{Z}$-flat, the Künneth formula shows that $P_* \otimes Q_*$ is actually a $\mathbb{Z}E$-resolution of $I \otimes J$.

Finally, $P_* \otimes Q_*$ is $\mathbb{Z}E$-projective. To see this it suffices to show that if M is a $\mathbb{Z}N$-projective $\mathbb{Z}E$-module, then $M \otimes \mathbb{Z}G$, qua $\mathbb{Z}E$-module by diagonal action, is $\mathbb{Z}E$-projective. This is true because

$$M \otimes \mathbb{Z}G \xrightarrow{\sim} (M\downarrow_N) \otimes_{\mathbb{Z}N} \mathbb{Z}E \quad \text{by } a \otimes e\pi \mapsto ae^{-1} \otimes e,$$

where π is the homomorphism $E \to G$. $\square$

There should be a companion to Theorem 5.5 dealing with silp. The best that we can do is the following result.

5.7 Theorem. *If* $1 \to N \to E \to G \to 1$ *is an exact sequence of groups, then* silp $E \leqslant$ silp $N +$ spli $G + 1$.

5.8 Lemma. (i) *If* silp $G = n$, *then induced modules have injective dimension* $\leqslant n$.

(ii) *If A is a $\mathbb{Z}$-free $\mathbb{Z}G$-module such that $A \otimes D$ has injective dimension $\leqslant n$ for every $\mathbb{Z}$-injective D, then A has $\mathbb{Z}G$-injective dimension $\leqslant n+1$.*

Proof. Part (i) is the dual of Lemma 5.6; part (ii) is a special case of Lemma 2.2. □

Proof of Theorem 5.7. Suppose silp $N = n$ and spli $G = m$ are finite. We must show that every $\mathbb{Z}E$-projective module has injective dimension $\leqslant m+n+1$. By Lemma 5.8 (ii) it suffices to show that, if $R = P \otimes D$, where D is $\mathbb{Z}$-injective and P is $\mathbb{Z}E$-projective, then R has $\mathbb{Z}E$-injective dimension $\leqslant m+n$. Applying Hom(, R) to the $\mathbb{Z}$-split sequence (7) gives the $\mathbb{Z}$-split sequence of $\mathbb{Z}G$-modules

$$0 \to \mathrm{Hom}(C, R) \to \mathrm{Hom}(J, R) \to R \to 0.$$

This is $\mathbb{Z}E$-split because R is a direct summand of an induced module and thus it will suffice to prove $\mathrm{Hom}(J, R)$ has injective dimension $\leqslant m+n$.

Take a $\mathbb{Z}E$-injective resolution of R and call I^* the truncation of this at n, say

$$0 \to R \to I^0 \to \cdots \to I^{n-1} \to I^n \to 0.$$

Since R has $\mathbb{Z}N$-injective dimension $\leqslant n$ (by Lemma 5.8 (i)), I^* is $\mathbb{Z}N$-injective.

If Q_* is a $\mathbb{Z}G$-projective resolution of J of length m, then $\mathrm{Hom}(Q_*, I^*)$ is a $\mathbb{Z}E$-complex over $\mathrm{Hom}(J, R)$ of dimension $m+n$. Now R is divisible as $\mathbb{Z}$-module, whence $\mathrm{Ext}^1_{\mathbb{Z}}(J, R) = 0$ and so Künneth's formula shows the complex $\mathrm{Hom}(Q_*, I^*)$ is acyclic in all positive dimensions.

To complete the proof we check that $\mathrm{Hom}(Q_*, I^*)$ is $\mathbb{Z}E$-injective. It suffices to show that $\mathrm{Hom}(\mathbb{Z}G, M)$ is $\mathbb{Z}E$-injective whenever M is a $\mathbb{Z}N$-injective $\mathbb{Z}E$-module. Let π be the projection $E \to G$ and define maps

$$\mathrm{Hom}(\mathbb{Z}G, M) \rightleftarrows \mathrm{Hom}_{\mathbb{Z}N}(\mathbb{Z}E, M \downarrow_N)$$

by

$$\varphi \mapsto (e \mapsto e^{-1} \pi\varphi e),$$
$$(e\pi \mapsto e^{-1}\psi e) \leftarrow\!\shortmid \psi.$$

These are inverse to each other and give $\mathbb{Z}E$-isomorphism when we view the left side as a module by diagonal action and the right side as a module by the usual coinduced action. (Cf. the proof of Lemma 2.3 for a more general version of this.) The right side is indeed $\mathbb{Z}E$-injective. □

An immediate consequence of Theorem 5.7 with $N = 1$ is the following Corollary.

5.9 Corollary. *If* spli G *is finite, then so is* silp G.

Of course, this corollary is a special case of our Theorem 2.4 on Hopf algebras. The proof of Theorem 5.7 when $N = 1$ is almost exactly our argument for Theorem 2.4 in case the Hopf algebra is $\mathbb{Z}G$.

5.10 Theorem. *Let* $1 \to N \to E \to G \to 1$ *be an exact sequence of groups and assume* $N \in \mathfrak{C}_p$, $G \in \mathfrak{C}_q$. *Then* $E \in \mathfrak{C}_{p+q}$. *Thus* $\mathfrak{C}$ *is extension closed.*

Proof. When $q = 0$ the result has been proved by Ikenaga using Serre's 'wreath product construction' [8, p. 450]. We now proceed by induction on q.

Let X be a G-complex of the correct type for q. Then E also acts on X and makes X an admissible E-complex. For each cell σ of X,

$$1 \to N \to E_\sigma \to G_\sigma \to 1$$

is exact and so, by the induction hypothesis, $E_\sigma \in \mathfrak{C}_{p+q-1}$. By 5.4, silp N = spli N and silp G = spli G are finite. Hence silp E is finite by Theorem 5.7 and silp E bounds silp E_σ (by 5.1). Therefore $E \in \mathfrak{C}_{p+q}$, as required. □

We may summarise our results as relations between group classes. Write SILP for the class of all groups G for which silp G is finite and define SPLI similarly. If $\mathfrak{X}, \mathfrak{Y}$ are group classes, then $\mathfrak{X}\mathfrak{Y}$ denotes, as is usual, the class of all extensions of $\mathfrak{X}$-groups by $\mathfrak{Y}$ groups (the normal subgroup in $\mathfrak{X}$, the factor group in $\mathfrak{Y}$). We have established the following:

$$\mathfrak{C} \subseteq \mathrm{SPLI} = (\mathrm{SPLI})^2 \subseteq \mathrm{SILP} = (\mathrm{SILP})(\mathrm{SPLI}).$$

We leave open whether the two inclusions are strict.

6. Soluble groups

To understand a new class of groups it is often helpful to study soluble groups in the class. We do this here with SILP and SPLI.

For any soluble group G, let $h(G)$ denote the sum of the torsion-free ranks of the derived factors of G:

$$h(G) = \sum_{n \geq 0} \dim_{\mathbb{Q}}(G^{(n)}/G^{(n+1)} \otimes_{\mathbb{Z}} \mathbb{Q}).$$

This is a non-negative integer or ∞ and is called the *Hirsch number* of G. The function h is additive on soluble groups in the sense that, when N is a normal subgroup of G, $h(G) = h(N) + h(G/N)$. Consequently $h(H) \leq h(G)$ for every subgroup H of G.

A weaker version of the following theorem was found independently by Ikenaga [12].

6.1 Theorem. *Let* G *be a soluble group and* T *the* (*unique*) *maximal normal torsion subgroup of* G. *Then*

$$h(G) \leq \mathrm{spli}\ G \leq 2 + h(G) + \mathrm{spli}\ T.$$

6.2 Corollary. *Assume* spli T *is finite. Then all of* $h(G)$, spli G, silp G *are finite if one of them is finite.*

Proof. By Theorem 6.1, $h(G)$ finite imples spli G finite, and by Corollary 5.9, spli G finite implies silp G finite.

It remains to prove silp G finite implies $h(G)$ finite. We do this by showing that $h(G)$ infinite implies silp G infinite. There is a first group in the sequence of derived factors

$$1, G^{(s-1)}, G^{(s-2)}/G^{(s-1)}, \ldots$$

having infinite Hirsch number. Say $h(G^{(k-1)}/G^{(k)})$ is infinite but $h(G^{(k)})$ is finite. For each positive integer n we can find a subgroup H containing $G^{(k)}$ so that $H/G^{(k)}$ is free abelian of rank n. Hence $n \leqslant h(H) < \infty$. The finiteness of $h(H)$ implies, as we already know, that spli H and silp H are finite. Using the first inequality of Theorem 6.1 on H,

$$n \leqslant h(H) \leqslant \operatorname{spli} H = \operatorname{silp} H \leqslant \operatorname{silp} G.$$

Since n was arbitrary, silp G is infinite. □

The hypothesis on T in the corollary is certainly satisfied whenever T, or equivalently G, has cardinality $\aleph_k$ for some finite k. For T is locally finite, whence the result is a simple consequence of a spectral sequence argument: cf. [8, Proposition 6].

The first inequality of Theorem 6.1 depends on the following two lemmas. We recall the definition of the weak dimension of a ring Λ. This is the largest among the integers k for which $\operatorname{Tor}_k^{\Lambda}(\ ,\ )$ is non-zero, or is ∞ if no such k exists.

6.3 Lemma. *If the ring* Λ *has weak dimension* m, *then* spli $\Lambda \geqslant m$.

Proof. Choose A, B so that $\operatorname{Tor}_m^{\Lambda}(A, B) \neq 0$ and let $0 \to A \to I \to L \to 0$ be an injective representation of A. The following segment of the long exact Tor sequence,

$$0 = \operatorname{Tor}_{m+1}^{\Lambda}(L, B) \to \operatorname{Tor}_m^{\Lambda}(A, B) \to \operatorname{Tor}_m^{\Lambda}(I, B)$$

shows that $\operatorname{Tor}_m^{\Lambda}(I, B) \neq 0$. Hence I cannot have a projective resolution of length $< m$ and thus spli $\Lambda \geqslant m$. □

6.4 Lemma. *For any group* G, spli $\mathbb{Q}G \leqslant$ spli $\mathbb{Z}G$.

Proof. Suppose spli $\mathbb{Z}G$ is finite, say m. Let I be an injective $\mathbb{Q}G$-module. Clearly I is $\mathbb{Z}G$-injective and hence has a projective $\mathbb{Z}G$-resolution P_* of length $\leqslant m$. Then $P_* \otimes \mathbb{Q}$ is a projective $\mathbb{Q}G$-resolution of $I \otimes \mathbb{Q} \cong I$. □

Proof of Theorem 6.1. For the first inequality we use a theorem of Stammbach [10] that $h(G)$ is the weak dimension of $\mathbb{Q}G$. This fact and Lemmas 6.3 and 6.4 give $h(G) \leqslant \text{spli}\, G$.

The second inequality will follow from the extension theorem on spli (Theorem 5.5) if we can show

$$\text{spli}\, G \leqslant 2 + h(G),$$

where now G denotes a soluble group without any (non-trivial) normal torsion subgroup. We shall prove G is of finite virtual cohomological dimension.

The Hirsch–Plotkin radical R of G is a torsion-free nilpotent group of finite Hirsch number, whence, by a result of Čarin, the automorphism group induced by G on R is isomorphic to a subgroup, say L, of $GL(n, \mathbb{Q})$ for some finite n ([3] or [11]). The solubility of G ensures that the kernel of the homomorphism $G \to L$ is the centre Z of R.

Now L contains a subgroup L_1 of finite index that is conjugate to a triangular group in $GL(n, F)$, where F is a suitable finite extension field of $\mathbb{Q}$. Hence L_1 contains a unipotent normal subgroup L_2 with L_1/L_2 isomorphic to a subgroup of the direct product of n copies of $F^{\times}$, the multiplicative group of F. Since the torsion subgroup of $F^{\times}$ is finite, so therefore is the torsion subgroup of L_1/L_2 and hence L_1/L_2 is the direct product of its torsion subgroup and a torsion-free group (e.g. [4, VII, 6.2]). Thus L has a torsion-free subgroup of finite index. It follows that G also has a torsion-free subgroup H of finite index. The cohomological dimension of H is at most $1 + h(H)$ [1, 7.10], whence (using also 5.2 and Corollary 1.7)

$$\text{spli}\, G = \text{spli}\, H = \text{gldim}\, H \leqslant 2 + h(H),$$

as required. □

Theorem 6.1 extends in a straightforward way to certain classes of generalised soluble groups.

We have stated in this paper rather more questions than we have answered and we have resisted the temptation to raise even more. It is worth pointing out however that there is also a theory of complete *homological* functors and *initial* completions. The various connexions between homological and cohomological completions are of considerable interest since they necessarily involve considerations of duality.

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